A conceptual problem for non-commutative inflation and the new approach for non-relativistic inflationary equation of state

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In a previous paper, we connected the phenomenological non-commutative inflation of Alexander, Brandenberger and Magueijo (2003, 2005 and 2007) with the formal representation theory of groups and algebras. In that paper, the fundamental equations of inflation followed as a consequence of a deformation of the Poincaré group, which induces a particular quantum representation. In this paper, we show that there exists a conceptual problem with the kind of representation that leads to the fundamental equations of the model and that the procedure to obtain those equations should be modified according to one of two possible proposals. One of them relates to the general theory of Hopf algebras. The other is based on a representation theorem of Von Neumann algebras, a proposal already suggested by us to take into account interactions in the inflationary equation of state. This reopens the problem of finding inflationary deformed dispersion relations and all developments which followed the first paper of Non-commutative Inflation.

I. INTRODUCTION

Non-commutative inflation, as originally conceived [1], is a phenomenological model of inflation inspired by varying speed of light cosmology [2], [3]. It was proposed as a possible mechanism connecting non-commutative spaces with inflation. In this model, radiation, rather than a scalar field, drives inflation. Non-commutative inflation is an application of a conjecture that the non-relativistic character of non-commutative spaces [4], [5], modeled by a non-relativistic dispersion relation, was enough to determine the thermodynamics of radiation.

Despite the original constraint with non-commutative spaces, non-commutative inflation is actually a bridge connecting general violations of Lorentz symmetry with inflation, an idea that can be related with a wider literature in high energy physics [6], [7], [8], [9] and [10]. Instead of the usual Lagrangian approach of physics, in which non-relativistic effects are modeled by non-relativistic terms that can affect the cosmological perturbations, such as in the so called trans-Planckian problem of inflation [11] and [12], the connection between non-commutative spaces and the fundamental equations of non-commutative inflation can be more directly related with the Wigner approach for QFT and the idea of Hopf algebra deformations of the Poincare Lie algebra due to non-commutative spaces [13], rather than the star product approach [14]. Although a general procedure to deform a Lie algebra as a function of non-commutative space information is not available in the literature, postulating the existence of a generalization of the analysis [13], it is possible to define the quantum theory of matter, not by a quantization of a deformed action assumed to be invariant by Hopf algebra, as previously done, or the star product approach to non-commutative field theories [14], but as a deformation of Wigner procedure to define quantum theory in terms of representations of the Poincaré group as quantum symmetries [15]. This procedure is assured, by the general techniques of algebra, like the GNS construction (see [16] and [17]), to have much in common with a Lie group representation, in particular, the fundamental role of the dispersion relation in the scheme.

The Hopf algebra concept, more than an effort to bridge non-commutative spaces with non-commutative inflation, is actually a key concept in a more rigorous formulation of non-commutative inflation, being a useful concept in group theory in terms of which one can ask what is the group (algebraic) structure related to inflation. Something similar to ask what is the algebraic structure related to a fundamental invariant length scale and a fundamental speed, leading to the seminal paper of the doubly special relativity [6] (see a generalization of this idea to include an invariant expansion rate [18]). More than that, the algebra is the key connecting a phenomenological deformed dispersion relation with the effective energy-momentum tensor of inflationary matter.

In this paper, we analyze the algebraic and the physical requirements for the group structure that realizes non-commutative inflation and what that group structure tell us about the energy-momentum tensor which describes matter in classical gravitational theory.

II. A GROUP THEORY ARGUMENT FOR NON-COMMUTATIVE INFLATION

The basic idea explored in [19] and [1] is that the effect of generic models of non-commutative space-time can be codified in the modification of the energy-momentum relation. This is indicated in most of the implementations

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of the non-commutative principle to field theory [4], [5]. This would affect the calculation of the canonical partition function for radiation that in turn affects the early phases of the universe in thermal equilibrium.

This idea is formalized in the Wigner approach to relativistic quantum theory, as already discussed in [15], in which the basic problem is to construct representations of the Poincaré group as quantum symmetries, not the quantization of a particular classical field. That is, finding the correspondence π among Lorentz transformations $x \to \Lambda x + a^\mu$, denoted by (Λ, a^μ), and the transformation $\pi[(\Lambda, a^\mu)]$: $\mathcal{H}_P \to \mathcal{H}_P$, that satisfies:

$$\pi[(\Lambda_1, a_1^\mu)] \cdot \pi[(\Lambda_2, a_2^\mu)] = \pi[(\Lambda_1, a_1^\mu) \cdot (\Lambda_2, a_2^\mu)], \quad (1)$$
$$\pi[(\Lambda_1, a_1^\mu)]^{-1} = \pi[(\Lambda_1, a_1^\mu)^{-1}], \quad (2)$$
$$\pi[(\Lambda, a^\mu)] \text{ is a quantum symmetry}, \quad (3)$$

where $\mathcal{H}_P$ is the projective Hilbert space, that is, the space of rays, $\cdot$ denotes the composition and $()^{-1}$ the inverse. The symmetry is characterized by the invariance of transition amplitudes:

$$|(T\Phi, T\Psi)| = |(\Phi, \Psi)|. \quad (4)$$

According to the Wigner theorem, a quantum symmetry can be extended from rays in $\mathcal{H}_P$ describing quantum states to the entire Hilbert space $\mathcal{H}$ as a linear unitary or a antilinear antiunitary transformation. After such an extension, we might be dealing with projective representations. i.e. eq.(1) might be valid except for a phase $e^{i\theta}$;

Implicitly in the Wigner prescription to quantum symmetries is that all transformations act in the same physical space that are left invariant by them and all transformations have an inverse, therefore form a group. That is, despite of all the possible complex algebraic structures that one may consider, we are always dealing with groups.

The Poincaré group P is a very particular group, a Lie group, that is, a group with topological structure. In addition, a Lie group which possess a connected subgroup, the proper and orthochronous part $P_+^\uparrow$ (for which the Lorentz subgroup satisfies $\Lambda^0{}_0 > 0$ and $\det\Lambda = 1$). The algebraic structure of the group, codified in the multiplication rule, does not constrain its topology and we could consider a particularly useful representative of this group, the so called universal enveloping group, in which the topology is simply connected, that is, every path between two point in the group space can be deformed continuously in any other. For the Poincaré group, the group is $SL(2, \mathcal{C})$. For this group, one can always adjust phases in every $\pi[(\Lambda_1, a_1^\mu)]$ in such a way that it will be a non-projective representation.

For connected Lie groups, the symmetry representation problem reduces to constructinf unitary representations, since, at least in the neighborhood of the identity described by canonical coordinates, every group element is part of a one-parameter subgroup and thus it is the square of some other element. In addition, every element of a connected Lie group can be written as a product of elements of an arbitrarilly small neighborhood of identity. The square of an antilinear antiunitary operator is linear and unitary. The continuous[1] one parameter family of unitary (non-projective) transformations affords a further simplification due, to the Stone theorem, that can be put in the form $e^{-iH\lambda}$, H a selfadjoint operator. Therefore, the information about the group, in the connected Lie group case, can be cast in terms of information about a set of generators H_i of one parameter subgroups of the canonical coordinates in the neighborhood of identity.

The set of generators form a linear Lie algebra:

$$[P^\mu, P^\nu] = 0 \quad (5)$$
$$[M^{\mu\nu}, P^\lambda] = i\left(g^{\mu\nu} P^\lambda - g^{\lambda\nu} P^\mu\right) \quad (6)$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(g^{\mu\rho} M^{\nu\rho} - g^{\nu\rho} M^{\mu\sigma} + g^{\nu\sigma} M^{\mu\rho} - g^{\mu\sigma} M^{\nu\rho}\right) \quad (7)$$

A unitary representation of a group can be further simplified by a change of basis, in such a way that all matrices $\pi[(\Lambda_1, a_1^\mu)]$ can be put in the block diagonal form. If this process cannot be continued, every block is an example of the simplest kind of representation, the so called irreducible representation. The unitary representation of a group summarizes into constructing the irreducible pieces by which every other representation can be constructed by. These basic parts are identified with the Hilbert space of one particle states. The dispersion relation $\mathcal{C}(p)$ is the fundamental information in this process because it is a (self-adjoint) function of space and time translation generators which commutes with all other generators of the symmetry group (Casimir of the Lie algebra) and defines a bounded operator [2] $(e^{iC(p)})$ which commutes with every element of the group. By an infinite dimensional version of Schur's lemma, a unitary representation of a group is irreducible if, and only if, every bounded operator which commutes with every element of the group is a multiple of the identity. Indeed, for the Poincaré group, the algebraically independent set of Casimir operators is:

$$\begin{gathered} P^2 = H^2 - \vec{P}^2, \\ W = -w^2, \text{ com } w_\rho = \frac{1}{2}\epsilon_{\alpha\mu\nu\rho} P^\alpha M^{\mu\nu}, \\ sign(P^0) = \theta(P^2)\epsilon(P^0), \end{gathered} \quad (8)$$

Given two irreducible representations of $SL(2, \mathcal{C})$, U_1 and U_2 in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively (with possible additional symmetries such as parity in the electrodynamics case), we can construct another representation in the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$

[1] i.e. $\langle\Psi|\, U(\lambda)\, |\Psi\rangle$ continuous for every Φ in the Hilbert space $\mathcal{H}$
[2] An operator A is bounded if $||A\Psi|| \leq C||\Psi||$, with C independent of Ψ

which describes two non-interacting relativistic particles:

$$U_1 \otimes U_2(\sum_{ij} c_{ij}\Psi_i \otimes \Phi_j) = \sum_{ij} c_{ij}(U_1\Psi_i) \otimes (U_2\Phi_j). \quad (9)$$

The generators of the Lie algebra are given by:

$$X_i^{\otimes} = X_i^{(1)} \otimes I + I \otimes X_i^{(2)}, \quad (10)$$

$X_i^{(i)}$ generators of the Lie algebra of the representation U_i.

In particular, we can make $U_1 = U_2$ and take into account indistinguishable particles by introducing a Hilbert space basis in the symmetrized form:

$$S_N(\psi_1 \otimes \cdots \otimes \psi_N) = \frac{1}{N!}\sum_{\sigma} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)}, \quad \psi_i \in \mathcal{H} \quad (11)$$

where σ is an index permutation, or in the antisymmetrized for:

$$A_N(\psi_1 \otimes \cdots \otimes \psi_N) = \frac{1}{N!}\sum_{\sigma} sign(\sigma)\psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)}, \quad (12)$$

where $sign(\sigma)$ is 1 for an even permutation and -1 for an odd permutation. This defines two kinds of Hilbert spaces:

$$\mathcal{H}_S^N = \{\psi \in \mathcal{H}^N; S_N\psi = \psi\} \quad (13)$$

and

$$\mathcal{H}_A^N = \{\psi \in \mathcal{H}^N; A_N\psi = \psi\}. \quad (14)$$

$\mathcal{H}^N$ denoting the tensor product of N Hilbert spaces.

We then define the Fock space of radiation:

$$U_F = \sum_{N=0}^{\infty}{}^{\oplus} \left(\mathcal{U}_{\lambda}^{\otimes N}\right)_S(\mathcal{P}) \quad (15)$$

where $\mathcal{U}_{\lambda}^{\otimes N}$ is the tensor product of N photon irreducible representations characterized by the set of eingenvalues λ, in particular $H^2 - P^2 = 0$. U_F is defined in the Hilbert space:

$$\mathcal{H} = \sum_{N=0}^{\infty}{}^{\oplus} \left(\mathcal{H}_{S\ \lambda}^N\right) \quad (16)$$

$\mathcal{H}_{S\ \lambda}^N$ being the symmetrized tensor product of the Hilbert spaces where the photon irreducible representations acts. $N = 0$ corresponds to the trivial vacuum representation $\pi[(\Lambda_1, a_1^{\mu})] = I$.

The associated Hamiltonian is:

$$H_F = \sum_{N=1}^{\infty}{}^{\oplus}(H \otimes I \cdots \otimes I + I \otimes H \otimes \cdots \otimes I + \cdots + I \otimes I \cdots \otimes H)_N, \quad (17)$$

In the irreducible representation, H can be put a form in which the momentum P^j is diagonal. Being an irreducible representation, all we need specify is the dimension d of the eigenspace associated with the eigenvalue p^j that can be proved to be the same for all p^j in irreducible representations:

$$H = diag_d(E(p), E(p), \cdots E(p)), \quad (18)$$

that denotes a $d \times d$ diagonal matrix, $E(p)$ being the photon dispersion relation. The dimension d is determined by the dimension of the corresponding little group, that is, the group subspace that leaves the four-momentum of the irreducible representation invariant. For $P^2 = 0$ and $P^{\mu} \neq 0$, we can choose the four-momentum $k = (1, 0, 0, 1)$ and using the correspondence between the Lorentz group action and $SL(2, \mathbb{C})$ action on 2×2 hermitian matrix, in such a way that k corresponds to the matrix $\bar{p}$ given by:

$$\bar{p} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (19)$$

that is leaved invariant by the action of the elements $A \in SL(2, \mathbb{C})$ given by ($\bar{p}' = A\bar{p}A^*$):

$$\gamma_{\phi} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \quad \gamma_{\eta} = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, \quad (20)$$

γ_{ϕ}, ϕ real (mod 2π), is a representation of the rotation group in two dimensions $SO(2)$, while γ_{η}, η is complex, the translation group. The associated little group is therefore $ISO(2)$ of the rotations and translations in $\mathbb{R}^2$; This subgroup is associated with translation and only admits infinite dimensional representations (except the trivial one) associated with the continous spin representations. Since we do not have knowledge of infinite internal degrees of freedom for particles, we postulate that this subgroup is only mapped into the trivial representation (all elements go into the identity). The $SO(2)$ subgroup has one dimensional irreducible representations, but since the electrodynamics is invariant by parity, that connects opposite helicities, the dimension of the little group is two.

The canonical partition function is then given by:

$$Z(\beta, V) = Tr\left(e^{-\beta H_F}\right) \quad (21)$$

In order to determine the dependence with volume, we impose periodic boundary conditions on the one parameter subgroups of space translations in the x^j direction:

$$U_{x^j}(0) = U_{x^j}(L). \quad (22)$$

It follows that:

$$U(x_i) = \sum_n e^{-i\frac{2\pi n x_i}{L}} \int_0^L \frac{1}{L} e^{i\frac{2\pi n y}{L}} U(y)dy = \sum_n e^{-i\frac{2\pi n x_i}{L}} E_n, \quad (23)$$

where E_n are mutually orthogonal projections and:

$$\lim_{n\to\infty}\sum_n E_n\Psi = \Psi. \tag{24}$$

This implies momentum quantization, since $U(t) = e^{-i\frac{P^i x^i}{\hbar}}$, where

$$P = \sum_n nE_n \tag{25}$$

The allowed values of momentum are:

$$p_j = \frac{2\pi\hbar n_j}{L}, \tag{26}$$

where n_j an integer.

The idea of non-commutative inflation [20] is to introduce the fenomenological modification

$$E^2 = P^2 \to E^2 = P^2 f^2(E) \tag{27}$$

straight into (18) and (17), modifying the calculation of (21) and obtaining, from the usual thermodynamic relations:

$$\rho(E,T) = \frac{1}{\pi^2}\frac{E^3}{\exp E/T - 1}\frac{1}{f^3}\left|1 - \frac{Ef'}{f}\right| \tag{28}$$

$$p = \frac{1}{\beta}\frac{\partial}{\partial V}ln\left(Z(\beta,V)\right) = \frac{1}{3}\int\frac{\rho(E,T)}{1-\frac{Ef'}{f}}dE \tag{29}$$

$$\rho = -\frac{1}{V}\frac{\partial}{\partial\beta}ln\left(Z(\beta,V)\right) = \int\rho(E,T)dE. \tag{30}$$

We consider here $c = k_B = \hbar = 1$. To find an inflationary behavior, [1] used the ansatz:

$$f = 1 + (\lambda E)^\alpha, \tag{31}$$

We extended this to a set of inequalities on $f(E)$ related to minimal requirements for inflation [21]. We use this extension as a starting point of our argument.

III. THE HOPF ALGEBRA CONCEPT

A question addressed by us in [15] is wheter the above analysis follows unaffected under the non-commutative hypothesis, since this model is meant to be a mechanism for inflation applicable to non-commutative spaces. Since what is meant by a quantum theory in the non-commutative space is a matter of discussion in the literature[22], [23], [24], [13] and [25], we followed a suggestion, more adjustable to the Wigner approach, that the effect of non-commutative space-time is to change the algebraic structure of the Poincaré Lie algebra [13]. This is something similar to quantizing a group, in which the algebraic structure of the Lie algebra is transformed into that of a Hopf algebra.

The information which defines the non-commutative space is the C^*-algebra [3], that, in some sense, is the algebraic idealization of a set of complex continuous functions defined on a topological space X. An important result, called the Gelfand-Naimark theorem, states that, when the product of the C^*-algebra is commutative, it is possible to recover the space X as a function of the algebraic information alone. Allowing the product of the algebra to be non-commutative, we can establish a non-unique correspondence with a set of Hilbert space operators. C^*-algebra is what one can call the flat non-commutative space, since the metric is not recovered by the algebraic information in this formalism, a generalization that is the fundamental concept in the non-commutative geometry is the spectral triple, [25], [26],[27],[28],[29] and [30], in which additional structure is furnished for the C^*-algebra to codify a metric.

Although the procedure to take into account the information that defines the non-commutative space, the C^*-algebra, and deform the Lie algebra structure is not defined for general non-commutative spaces, if we assume that such a procedure must exist, we are able to define a formal deformation of the relativistic quantum theory in terms of algebraic methods [15]. By doing this, we are following a different approach of the very authors of the Hopf algebra deformation procedure, in which, once possessing the associated Hopf algebra, they define the physics (only for real free scalar field) by a Lagrangian action to be quantized. A Lagrangian that is required to be invariant under Hopf Algebra action:

$$S[\phi(x)] = \int \phi(x)(\Box_\lambda - M^2)\phi(x), \tag{32}$$

where $\Box_\lambda$ is a non-relativistic differential operator.

The Hopf algebra concept, instead of a formal need for a possible connection with non-commutative spaces, is actually a fundamental ingredient of non-commutative inflation.

There are many ways to define a Hopf algebra, a possible way which suggests its connections with non-commutative spaces is to tell that Hopf algebra is a set of transformations which acts not only on a vector space (like the Hilbert space), but on a algebra, like the C^*-

[3] C^*-algebra is a linear vector space $\mathcal{A}$ with an associative product $\cdot : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ (i.e. $(a\cdot b)\cdot c = a\cdot(b\cdot c)$); an operation called involution $* : \mathcal{A}\to\mathcal{A}$ that is defined with the properties: $(A+B)^* = A^* + B^*$, $(\lambda A)^* = \overline{\lambda}A^*$, with λ a complex number, $(AB)^* = B^*A^*$ and $(A^*)^* = A$; a norm $||\ || : \mathcal{A}\to\Re$ with respect to which the algebra is a Banach space (i.e. given a sequence a_n of elements, if $\lim_{n\to\infty}||a_{n+m} - a_n|| = 0$ for each $m > 0$, then there exists an a such that $\lim_{n\to\infty}||a_n - a|| = 0$.); the product is continuous with respect to the norm, i.e. $||AB|| \leq ||A||\cdot||B||$, and the norm additionally satisfies $||a^*a|| = ||a||^2$

algebra, and acts in a way dependent on the algebra product, that codifies non-commutativity.

A Hopf algebra H acts in a algebra $\mathcal{A}$ as a set of transformations not necessarily invertible, $H : \mathcal{A} \to \mathcal{A}$, being this action denoted by $h \triangleright f$, $h \in H$ and $f \in \mathcal{A}$. H is an algebra, that is, one can multiply elements, and construct linear combinations with complex coefficients:

$$(h_1 \cdot h_2) \triangleright f = h_1 \triangleright (h_2 \triangleright g) \tag{33}$$
$$(\alpha h_1 + \beta h_2) \triangleright f = \alpha(h_1 \triangleright f) + \beta(h_2 \triangleright f). \tag{34}$$

In this way, H depends on the product of the elements of $\mathcal{A}$, given by a generalization of Leibniz rule: $h \triangleright (f \cdot g) = \sum_i (h_{i(1)} \triangleright f) \cdot (h_{i(2)} \triangleright g)$, or, by using a short notation:

$$h \triangleright (f \cdot g) = (h_{(1)} \triangleright f) \cdot (h_{(2)} \triangleright g), \tag{35}$$

where the rule

$$\Delta : H \to H \otimes H \tag{36}$$

given by $\Delta : h \to \sum_i h_{i(1)} \otimes h_{i(2)}$ is called a ***coprodut***.

It leads us to an alternative way to define a Hopf algebra, a way which is particularly important for the purposes in this paper. A Hopf algebra H is an algebra which has a rule to construct tensor products that furnishes representations of the same algebra.

Consider for example a Lie Algebra realized in the Hilbert space $\mathcal{H}$. Its generators satisfy:

$$[X_i, X_j] = C^k_{ij} X_k \tag{37}$$

But we can define new representations of the very same Lie algebra (37) in the tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_1$ by:

$$\Delta X_i = X_i \otimes I + I \otimes X_i \tag{38}$$

we observe that this is exactly what is done in (17).

We can actually construct arbitrary tensor products, because Δ has a property called ***coassociativity***:

$$(id \otimes \Delta)\Delta h = (\Delta \otimes id)\Delta h \tag{39}$$

$$\begin{tikzcd}
& h_{(1)} \otimes h_{(2)} \arrow[dr, "id\otimes\Delta"] & \\
h \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \\
& h_{(1)} \otimes h_{(2)} \arrow[ur, "\Delta\otimes id"'] &
\end{tikzcd}$$

Here id is the indentity among the operators $H \to H$. This follows from $H \triangleright (abc)$, where we apply the rule (35) first on the pair $a \cdot (bc)$, but it must yield the same result as applied first on the pair $(ab) \cdot c$.

A generalization of rule (35) for the product of N elements of $\mathcal{A}$ follows from coassociativity:

$$H \triangleright (\prod_i f_i) = \prod_i (h_{(i)} \triangleright f_i), \tag{40}$$

In addition, we can define a generalization of the rule (36) $\Delta^N : H \to H^{\otimes N+1}$:

$$\Delta^N h = (\Delta \otimes id \cdots \otimes id) \cdots (\Delta \otimes id \otimes id)(\Delta \otimes id)\Delta h, \tag{41}$$

which yields, by coassociativity, the same result, even if in each term $(\Delta \otimes id \cdots \otimes id)$ we would change the positions of the Δ. In this sense, we can identify the Δ itself with $(\Delta \otimes id \cdots \otimes id)$ while acting in $H^{\otimes N}$.

If we have a rule $\Delta : H^{\otimes N} \to H^{\otimes N+1}$, we must have a rule to do the inverse. This rule is obtained when one of the multiplying elements of $\mathcal{A}$, in the product rule (35), is the identity of $\mathcal{A}$:

$$\begin{aligned}
h \triangleright (1 \cdot g) &= (h_{(1)} \triangleright 1) \cdot (h_{(2)} \triangleright g) \\
&= \epsilon(h_{(1)}) \cdot (h_{(2)} \triangleright g) \\
&= h \triangleright g; \\
h \triangleright (g \cdot 1) &= (h_{(1)} \triangleright g) \cdot (h_{(2)} \triangleright 1) \\
&= (h_{(1)} \triangleright g) \cdot \epsilon(h_{(2)}) \\
&= h \triangleright g;
\end{aligned}$$

where the rule $\epsilon : H \to \mathbb{C}$ is called the ***counity***. The counity undoes the Δ action, in the following sense:

$$\begin{tikzcd}
& \epsilon(h_{(1)}) \otimes h_{(2)} \arrow[dr] & \\
\Delta h = h_{(1)} \otimes h_{(2)} \arrow[ur, "\epsilon\otimes id"] \arrow[dr, "id\otimes\epsilon"'] & & h \\
& h_{(1)} \otimes \epsilon(h_{(2)}) \arrow[ur] &
\end{tikzcd}$$

We therefore set $\epsilon : H^{\otimes N+1} \to H^{\otimes N}$, where ϵ is identified with $(\epsilon \otimes id \cdots \otimes id)$, independent of the position of ϵ.

The Hopf algebra H acts in $\mathcal{A}$ as a set of transformations not necessarily invertible.In spite of that, there is a generalized notion of inverse called antipode, $S : H \to H$. The antipode is such that if H has an inverse, it will satisfy $S(h) = h^{-1}$. The antipode is (uniquely) defined by the properties $S(h_{(1)}) \cdot h_{(2)} = h_{(1)} \cdot S(h_{(2)}) = \epsilon(h) 1_H$, where 1_H is the identity of H. In a short notation:

$$\cdot (S \otimes id)\Delta h = \cdot (id \otimes S)\Delta h = \epsilon(h) 1_H. \tag{42}$$

Given $h_{(1)} \otimes \cdots \otimes h_{(N+1)} = \Delta^N h$, we apply $\cdot(S \otimes id) \otimes id \cdots \otimes id$, possibly changing the position of the term $\cdot(S \otimes id)$, which yields $\Delta^{N-2} h$.

IV. CONNECTING HOPF ALGEBRAS WITH THE HILBERT SPACE

As already observed, in [13], the authors suggest that, once possessing a Hopf algebra which replaces the Poincaré Lie algebra, they define the physics (only for a real free scalar field) by an action functional to be

quantized. An action functional that is required to be invariant under Hopf Algebra action:

$$S[h \triangleright \phi(x)] = S[\phi(x)], \tag{43}$$

where $\phi(x)$ is an element of the non-commutative C^*-algebra which defines the non-commutative space. In our work, [15], we suggest an alternative formulation. This formulation consist in applying the Wigner prescription for relativistic quantum theory by assigning irreducible representations of the Hopf algebra to Hilbert spaces of one particle systems. A useful concept here is the GNS construction of the mathematical C^* algebra theory.

The GNS construction is of fundamental importance in physics and mathematics. It is the basis of the proof of the Von Neumann theorem, which states that all irreducible representations of the Heisenberg algebra are unitarily equivalent. The GNS construction is what assures us that the representation problem of C^* algebras always has a solution, provided that all algebraic conditions of its definition are satisfied. It creates a correspondence between the algebra and a set of Hilbert space operators.

The fundamental idea is that for all state $\omega : \mathcal{A} \to \mathbb{C}$, i.e. positive definite linear functional, defined by $\omega(A^*A) \geq 0$, corresponds a representation $\pi_\omega : \mathcal{A} \to \mathcal{O}(\mathcal{H}_\omega)$, $\mathcal{O}(\mathcal{H}_\omega)$ being operators in the Hilbert space $\mathcal{H}_\omega$. It is done in such a way that $\mathcal{H}_\omega$ has a cyclic vector Ψ_ω, that is, the application, in Ψ_ω, of the operators of the representation, generates the entire Hilbert space (more exactly, a dense subspace), in such a way that $\omega(A) = (\Psi_\omega, \pi_\omega(A)\Psi_\omega)$. Furthermore, any other representation π in Hilbert space $\mathcal{H}_\pi$ with a cyclic vector Ψ which satisfies $\omega(A) = (\Psi, \pi(A)\Psi)$ is unitarily equivalent to π_ω, that is, there exists an unitary transformation $U : \mathcal{H}_\pi \to \mathcal{H}_\omega$ such that:

$$U\pi(A)U^{-1} = \pi_\omega(A), \quad U\Psi = \Psi_\omega. \tag{44}$$

The GNS construction is based on the fact that the C^* algebra is already an Hilbert space, except for a scalar product to be defined. This can be taken as $(A, B) = \omega(A^*B)$, where ω is a state. The inner product is positive definite by positivity of the state, but $(A, A) = 0$ does not imply in general $A = 0$, an inner product property. We define then the set:

$$J = \{A \in \mathcal{A}, \omega(B^*A) = 0, \forall B \in \mathcal{A}\} \tag{45}$$

that is such that $\mathcal{A} \cdot J \subseteq J$, property which makes J a ***left ideal***. The states of the Hilbert space are then equivalence classes defined in terms of J, that is, any two elements that differ by a J element are equivalent: $[A] = A + B, B \in J$, which defines the so called quotient space $\mathcal{A}/J$. The Hilbert space operators are then given by:

$$\pi_\omega(A)[B] = [AB]. \tag{46}$$

and such that:

$$\omega(A) = \langle \Psi_\omega | A | \Psi_\omega \rangle \tag{47}$$

The existence of representations is therefore assured by the existence of the states, positive definite linear functionals, but it is part of the C^* algebra theory to prove that such states do exist, provided that the algebraic properties of the definition be satisfied.

Furthermore, different kinds of states are related to different kinds of representations. In particular, the irreducible representations are related to normalized states (i.e. $\omega(1) = 1$) that cannot be decomposed as convex linear combinations of any other two normalized states, i.e., $\omega(A) \neq \lambda\omega_1(A) + (1 - \lambda)\omega_2(A)$, $\lambda \in [0, 1]$ and ω_i an state, $\omega_i \neq \omega$. In particular, Schur's lemma is valid, and a state related to irreducible representation satisfies $\omega(P(C)) = P(\omega(C))$, where C is a Casimir of the algebra and $P(C)$ in a polynomial in C.

We can do the same thing with Hopf algebras. There exists many ways in which a Hopf algebra can act on other algebraic structures and many ways to realize its action. The particular case given by Eq. (35) is called a left corregular action and it is the particular way explored in [13] to define the Poincaré Lie algebra deformation under non-commutative hypothesis. In group theory, for example, the left multiplication of an element g by another element g' furnishes a representation of the group: $\pi[g](g') = g \cdot g'$, but it is not the case in Hopf algebras. To do that we must consider the linear functionals on it, the so callled dual Hopf algebra, $H^* : H \to \mathbb{C}$, denoted by $\langle \phi, h \rangle \to \mathbb{C}$, $\phi \in H^*$, $h \in H$. We can transform H^* into a Hopf algebra through the structure of H: $\langle \phi\psi, h \rangle = \langle \phi \otimes \psi, \Delta h \rangle$, $\langle \Delta\psi, g \otimes h \rangle = \langle \psi, g \cdot h \rangle$, $\langle 1, h \rangle = \epsilon(h)$ and $\epsilon(\psi) = \langle \psi, 1 \rangle$.

We then define the action of H into H^* according to (35), which transforms H^* into the so called H-module algebra, yielding the so called left corregular representation $R^* : H^* \to H^*$:

$$\left\langle R^*_g(\phi), h \right\rangle =< \phi, hg >=< \phi_{(1)}, h > \cdot < \phi_{(2)}, g > . \tag{48}$$

To proceed in the GNS construction, we need to define a involution $* : H \to H$, the adjoint operation of C^* algebras, characterized by the properties $(A+B)^* = A^* + B^*$, $(\lambda A)^* = \overline{\lambda} A^*$ such that $(AB)^* = B^*A^*$ and $(A^*)^* = A$. We need actually to assure that the self adjoint property $X = X^*$ is compatible with the algebra in such a way that we can transform it into generators of one parameter subgroups. Furthermore, we need compatibility with the Hopf algebra structure in the following sense: $\Delta h^* = (\Delta h)^{*\otimes*}$, $\epsilon(h^*) = \overline{\epsilon(h)}$ and $(S \circ *)^2 = id$.

As in the C^* algebra case, H^* is a Hilbert space, except for a inner product to be defined. This product must be such that $(\phi, h \triangleright \psi) = (h^* \triangleright \phi, \psi)$ $\phi, \psi \in H^*$. The Hopf algebra theory yields a natural choice of inner product, called the right integral $\int : H^* \to \mathbb{C}$ defined by the property:

$$(\int \otimes id) \circ \Delta = 1_{H^*} \cdot \int . \tag{49}$$

Analogously we can define the left integral, both being unique, except by a multiplicative factor.

The importance of the right integral is how it is related to the left corregular action:

$$\int (\phi \triangleright g)^* f = \int g^* (\phi^* \triangleright f) \tag{50}$$

We can define the ideal J_{H^*}:

$$J_{H^*} = \{ f \in H^* / \int g^* f = 0, \forall g \in H^* \} \tag{51}$$

that defines the Hilbert space H^*/J_{H^*}. It follows that we transform the Hilbert space into a non-commutative algebra in which the Hopf algebra acts following the rule (35), the same rule as in it acts on the non-commutative C^*-algebra.

We could, however, interpret the Hopf algebra simply as an algebra which can be realized as a C^*-algebra, by functionals that act on H itself, not in H^*, that is, ignoring the rule (35) and worrying about the coproduct only if we have to deal with tensor products. This is going to be the point of view of the rest of this work.

The prescription to define the physics of the radiation that would lead to the same equations of the non-commutative inflation, (28) and (29), would be:

Prescription 1 ($\pi_{\mathfrak{g}}$). *Be $\mathfrak{p}_{\mathcal{N}}$ a Hopf algebra with generators X_i which deforms Poincaré Lie algebra [4] with a set of Casimir elements $C_0, C_1 \cdots$ and be $\pi_{\lambda'} : \mathfrak{p}_{\mathcal{N}} \to O(\mathcal{D})$ the irreducible representation [5] with Casimir eigenvalues $\lambda' = (\lambda_0, \lambda_i)$. Be $\mathcal{P}_{\mathcal{N}}$ the group generated by elements of the form $e^{-iX_i t}$ and $\mathcal{U}^{\mathcal{NP}}_{\lambda'}$ an irreducible representation of it given by $\pi_{\mathcal{P}_{\mathcal{N}}}[e^{-iX_i t}] = e^{-i\pi_{\lambda'}(X_i)t}$, $t \in \mathbb{R}$. The physics of the radiation is defined by a representation of $\mathcal{P}_{\mathcal{N}}$ obtained by (15) by the replacement:*

$$\mathcal{U}_\lambda \to \mathcal{U}^{\mathcal{NP}}_{\lambda'}, \tag{52}$$

for some set λ'. This prescription leads us to the following representation of $\mathcal{P}_{\mathcal{N}}$:

$$U^{\mathcal{NP}}_F = \sum_{N=0}^{\infty} {}^{\oplus} \left(\mathcal{U}^{\mathcal{NP} \otimes N}_{\lambda'} \right)_S \tag{53}$$

Observe that the role of the GNS construction, if not directly applied to realize the algebra, is tells us that, once we define a C^* (or Hopf) algebra with all algebraic requirements, it is assured that the representation problem has a solution and we are able to do definitions like the above one. This suggests to us to define the group by definig a C^* algebra with appropriate requirements for its realization.

V. IMPORTANT PHYSICAL REQUIREMENTS ON THE POINCARÉ LIE ALGEBRA DEFORMATION $\mathfrak{p}_{\mathcal{N}}$ AND ON THE REPRESENTATION OF THE ASSOCIATED GROUP $\mathcal{P}_{\mathcal{N}}$ THAT DRIVES INFLATION

A. Abstract algebraic conditions

We thus define a deformation of the Poincaré Lie group $\mathcal{P}$ through a deformation of its universal enveloping algebra [6] $\mathfrak{p}$ to select an representation of it to describe the radiation and drive the inflation. This deformation, denoted by $\mathfrak{p}_{\mathcal{N}}$, besides the conditions to assure the existence of its realizations, must satisfy some physical requirements that we now describe.

First is that $\mathfrak{p}_{\mathcal{N}}$ must form a nontrivial algebra, that is, its generators must satisfy a set of algebraic equations not trivially satisfied by any set of Hilbert space operators. It may appear a trivial condition, but it is in terms of those nontrivial algebraic equations that the physical requirements are expressed. On the other hand, it specifies the algebraic structure of $\mathfrak{p}_{\mathcal{N}}$. Furthermore, the generators of this algebra and the algebraic constraints must be compatible with the self adjoint condition $X = X^*$, for an involution satisfying $(A + B)^* = A^* + B^*$, $(\lambda A)^* = \overline{\lambda} A^*$ and such that $(AB)^* = B^* A^*$.

$\mathfrak{p}_{\mathcal{N}}$ must have a commutative subalgebra associated to space-time translation generators, that we will associate with the energy E and momentum P, and a Casimir $C(E, P)$ obtained as an algebraic function of E, P that is compatible with the positive energy condition. Such compatibility is in the following sense: The solutions of the equation $C(E, P) = \lambda$ define surfaces in the (E, P) variables, that we call a ***mass-shell deformation***, denoted by $\mathcal{C}_\lambda$, such that for some subset $M_{\mathfrak{p}_{\mathcal{N}}}$ of λ, $\mathcal{C}_\lambda$ has a connected component satisfying $E \geq 0$ and $E = 0 \Leftrightarrow P^\mu = 0$. The set of the (E, P) points such that $C(E, P) \in M_{\mathfrak{p}_{\mathcal{N}}}$ and $E \geq 0$ we are going to call the ***deformed forward light cone*** $V^+_{\mathfrak{p}_{\mathcal{N}}}$. This property is part of the conditions that will assure that there exists irreducible representations of $\mathcal{P}_{\mathcal{N}}$ with positive-definite energy.

The algebra generated by energy and momentum, denoted by $\mathfrak{t}$, must be an ***ideal*** in the Lie algebra. That is, $[P^\mu, X_j] = G^\mu(P^\nu)$, where $G^\mu(P^\nu)$ is an algebraic function of energy and momentum. This condition assures that energy eigenstates, related to stable states, are mapped into energy eingenstates by the action of $\mathcal{P}_{\mathcal{N}}$. That is, if the system is observed to be in a stable state in one reference frame, it will be observed to be in a stable state after the application of $\mathcal{P}_{\mathcal{N}}$ [7]. That the subalgebra

[4] More exactly, its universal enveloping algebra, where multiplication between elements is defined

[5] Associated to the GNS construction given by states in $\mathfrak{p}_{\mathcal{N}}$ or $\mathfrak{p}^*_{\mathcal{N}}$. $\mathcal{O}(\mathcal{D})$ denotes operators in the domain $\mathcal{D}$.

[6] The associative algebra of generators, that is, there exists an associative product such that the commutator is the Lie product

[7] Suppose Ψ is an energy eingenstate, since $[E, P] = 0$, it is also a momentum eigenstate. We can then write $P^\mu \Psi = p^\mu \Psi$, or, equivalently, $e^{iP^\mu} \Psi = e^{ip^\mu} \Psi$. A referential change by

$\mathfrak{t}$ is an ideal essentially assures that:

$$U(\mathfrak{g}_1)P^{\mu}U^{-1}(\mathfrak{g}_1) = G_{\mathfrak{g}_1}(P^{\mu}), \tag{54}$$

where $U(\mathfrak{g}_1)$ is a Hilbert Space representation of element $\mathfrak{g}_1 \in \mathcal{P}_{\mathcal{N}}$ and $G_{\mathfrak{g}_1}$ is a realization of the element $\mathfrak{g}_1$ of the group $\mathcal{P}_{\mathcal{N}}$ as transformation in the energy-momentum space.

From this, we formulate the additional condition that assures positivity of energy for at least some irreducible representations of $\mathcal{P}_{\mathcal{N}}$ [8] :

$$[P^{\mu}, X^{\nu}] = c_0^{(\mu\nu)} + c_{\alpha}^{(\mu\nu)}P^{\alpha} + c_{\alpha\beta}^{(\mu\nu)}P^{\alpha}P^{\beta} + \cdots, \quad c_0^{(\mu\nu)} = 0 \tag{55}$$

$c^{(\mu\nu)}$ being complex coeficients.

The next requirement is that $\mathfrak{p}_{\mathcal{N}}$ has the Poincaré Lie algebra as the low energy-momentum limit. A way to assure this is as follows: Let $C_k^{ij\,(0)}$ be the structure constants of the Poincaré Lie algebra (5)-(7), and $C_k^{ij}(E,P)$ algebraic functions of the energy and momentum operators such that $C_k^{ij}(0,0) = C_k^{ij\,(0)}$ (Since $[E,P]=0$, we can put the Hilbert space in a representation in which E and P are multiplicative operators). We can define $\mathfrak{p}_{\mathcal{N}}$ as:

$$[X_i, X_j] = iX^kC_k^{ij}(H,P). \tag{56}$$

It is possible to prove that in a irreducible representation satisfying (54) the states can be written as:

$$\Psi = \sum_{\sigma}\int d^3\mu(p)\phi(p^{\mu},\sigma)\Psi_{p^{\mu},\sigma}, \tag{57}$$

(The conclusion that follows from (54) is that the dimension of eigenspace of P^{μ} with eigenvalue p^{μ} is independent of p^{μ})

We write then a state in the form (57) as:

$$\Psi_{p^{\mu}_{\max}} = \int d\mu(p)\phi(p,\sigma)\Psi_{p,\sigma}, \quad \mathrm{supp}\{\phi(p,\sigma)\} \in \{E < E_{\max}, P^j < p^j_{\max}\} \tag{58}$$

The action of generators of $\mathfrak{p}_{\mathcal{N}}$ on those states satisfy:

$$[X^i, X^j] \triangleright \Psi_{p^{\mu}_{\max}} = iX^kC_k^{ij}(H,P) \triangleright \Psi_{p^{\mu}_{\max}} \to iX^kC_k^{ij\,(0)} \triangleright \Psi_{p^{\mu}_{\max}} \tag{59}$$

when $p^{\mu}_{\max} \to 0$ and $E_{\max} \to 0$. supp denotes the smallest closed set outside which $\phi(p)$ is zero. $\triangleright$ denotes the action (as an operator) in the one particle states, while $\to$ denotes the strong convergence (i.e. $||X^kC_k^{ij}(H,P) \triangleright \Psi - X^kC_k^{ij\,0} \triangleright \Psi|| \to 0$)[9] . In other words, the typical eigenvalues of the generators effectively change the commutation relations. The property (56) further assures that the algebra generated by E, P is an ideal.

Given the property (54), it makes sense to talk about the little group, the subgroup $\mathcal{W}_{p^{\mu}}$ of $\mathcal{P}_{\mathcal{N}}$, in representation (54), that leaves a particular p^{μ} invariant. The little group of any p^{μ} in the same $\mathcal{C}_{\lambda}$ is the same (isomorphic). Indeed, in a irreducible representation, any p^{μ} can be sent into any other by the action of some $G_{\mathfrak{g}_1}$. Therefore, given $w_1 \in \mathcal{W}_{p_1^{\mu}}$ we can obtain $w_2 \in \mathcal{W}_{p_2^{\mu}}$ by $w_2 = G_{\mathfrak{g}_1}w_1G_{\mathfrak{g}_1}^{-1}$, where $G_{\mathfrak{g}_1}(p_1^{\mu}) = p_2^{\mu}$. It follows that, the little group that acts in low energy and momentum should be the same as that at high energies. What leads us to the requirement that every little group of $\mathcal{P}_{\mathcal{N}}$, $\mathcal{W}_{p^{\mu}}$, for $\lambda \in M_{\mathcal{P}_{\mathcal{N}}}$ must be isomorphic to some little group $\mathcal{W}(\Lambda, p^{\mu})$ of the Poincaré group associated to some positive-definite energy irreducible representation. What additionally assures that the number of internal degrees of freedom for particles is the same as that of relativistic particles, which affects the calculation of the partition function, as in (19).

A further condition is that $\mathfrak{p}_{\mathcal{N}}$ must preserve the $SO(3)$ Lie algebra, the rotation group, and $[H, M_i] = 0$, where M_i is the generator of the rotation subgroup along the i direction. Indeed, it is related to the thermodynamic description of a perfect fluid that is used in noncommutative inflation. That description is no longer valid if the thermodynamic state is no longer rotationally invariant, since the perfect fluid is the one that is isotropic in the reference frame that follows the fluid. In the thermodynamic description, the matter is described by the non-pure state $\langle\Psi_T|\, A\, |\Psi_T\rangle = \frac{Tr(e^{-\beta H}A)}{Tr(e^{-\beta H})}$, where A is the selfadjoint operator related to an observable and $\beta = \frac{1}{k_BT}$. The state is rotation invariant if

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$\mathcal{P}_{\mathcal{N}}$ implies $e^{iP^{\mu}} \to Ue^{iP^{\mu}}U^{-1} = e^{iC}e^{iP^{\mu}}$, what implies that $e^{iC} = Ue^{iP^{\mu}}U^{-1}e^{-iP^{\mu}}$. We can write $U = e^{iX_i}$, and apply the integral form of the Baker-Hausdorff formula:

$$e^C = e^Ae^B$$

$$C = B + \int_0^1 g(e^{tadA}e^{adB})Adt; \quad g = ln(z)/(z-1),$$

where adA is the adjoint action given by $adA(B) = [A,B]$. We conclude that $C \in \mathfrak{t}$ and, furthermore, $e^{iC}e^{iP^{\mu}} = e^{i\bar{P}^{\mu}}$, where $\bar{P}^{\mu}$ is an analytic function P^{μ} that defines a representation.

[8] The measurement in such representations yields expected values for four-momentum $\omega(P^{\mu}) = \langle\Psi_{\omega}|\, P^{\mu}\, |\Psi_{\omega}\rangle$ belonging to the linear convex closure of the set $\mathcal{C}_{\lambda}$, that we denote $\overline{\mathcal{C}_{\lambda}}_c$, and is the set of all linear combinations of the type $\lambda p_1^{\mu} + (1-\lambda)p_2^{\mu}$, $p_{1,2}^{\mu} \in \mathcal{C}_{\lambda}$, $\lambda \in [0,1]$. Being a regular representation, that is, a representation in which $e^{-iX_it}\,|\Psi\rangle$ is continuous for every state $|\Psi\rangle$, $\omega_t(P^{\mu}) = \omega(e^{iXt}P^{\mu}e^{-iXt})$, it is a continuous path in $\overline{\mathcal{C}_{\lambda}}_c$ that only achieves the region with $E \leq 0$ by the point $p^{\mu} = 0$, or, equivalently by the state $\omega_{t_0}(P^{\mu}) = 0$, but $\frac{d}{dt}\omega_t(P^{\mu})\big|_{t=t_0} = \omega([X_{t_0}, P^{\mu}]) = \omega(H(P^{\mu})) = 0$, $X_{t_0} = e^{iXt_0}Xe^{-iXt_0}$, where we have made use of (55).

[9] For selfadjoint operators, or $||X^kC_k^{ij}(H,P) \triangleright \Psi - X^kC_k^{ij\,0} \triangleright \Psi|| \to 0$, or it is not a convergent sequence

$\langle\Psi_T|\, RAR^{-1}\,|\Psi_T\rangle = \langle\Psi_T|\, A\,|\Psi_T\rangle$, where R is a rotation. Since the trace is cyclic, we have:

$$\frac{Tr(e^{-\beta H}A)}{Tr(e^{-\beta H})} = \frac{Tr(R^{-1}e^{-\beta H}RA)}{Tr(e^{-\beta H})}, \forall A \Leftrightarrow [M_i, H] = 0 \tag{60}$$

B. Conditions on the quantum representation

A last condition is related to the application of prescription 1, (17) must be a generator of the $\mathfrak{p}_{\mathcal{N}}$ if $H \in \mathfrak{p}_{\mathcal{N}}$. Although this is a subtle point, it is the main point of this paper. (53) is actually a valid representation of the group $\mathcal{P}_{\mathcal{N}}$, since it satisfies (1)-(3), but, at least that it is a Lie Group, the generators of the one-parameter subgroups are not assured to form the same algebra as $\mathfrak{p}_{\mathcal{N}}$. If it is the case, it means that we do not have a uniquely defined notion of a dispersion relation that replaces the relativistic one. In particular, the generators of these subgroups may not form an algebra, not satisfying the above discussed requirements. If direct product of two one-particle representations do form an algebra satisfying all the above requirements, but do not form the same algebra than that of the one particle subspace, then an S matrix connecting a one-particle with the two particles would not be compatible with energy conservation in all reference frames, since the energy of in and out states would not transform as the same function.

This leads us to consider two kinds of representations of the group $\mathcal{P}_{\mathcal{N}}$, the one whose generators of one-parameters subgroups do not form the same algebra than $\mathfrak{p}_{\mathcal{N}}$, that we will call the $\pi_{\mathfrak{g}}$ representation, and that one whose generators of the one-parameter subgroups do form the same algebra as $\mathfrak{p}_{\mathcal{N}}$, that we will call the $\pi_{\mathcal{A}}$ representation.

VI. THE NON-EXISTENCE OF A LIE GROUP THAT REALIZES NON-COMMUTATIVE INFLATION

In our previous work [15], we obtained a set of conditions to be satisfied by a dispersion relation that assures minimal desirable conditions for inflation, like the minimum duration of it, related to problems like the flatness of the universe [31], [21]. These conditions were given by a set of inequalities that contains the particular ansatz applied by the original authors of non-commutative inflation [1]. It was shown that, given the original equations of the model, (28)-(30), the minimum duration of inflation is related to a limitation of the photon's momentum (a consequence of condition 3 of the theorem 2 in [15]).

Since Casimir is an algebraic entity of $\mathfrak{p}_{\mathcal{N}}$, those conditions should be included in the set of algebraic conditions for inflation discussed in the last section. Let us show here that the limitation of the photon's momentum is precisely what implies that (17) cannot be the time translation generator of a representation of $\mathfrak{p}_{\mathcal{N}}$, provided that all C_λ (the mass shell deformation) with positive definite energy is connected and allows a zero momentum state, that is, C_λ contains the point $(E, P) = (E^{(0)}_\lambda, 0)$.

Consider the deformed forward light cone $V^+_{\mathfrak{p}_{\mathcal{N}}}$ defined in the previous section. Suppose that $C(E, P) = \lambda_P \in M_{\mathfrak{p}_{\mathcal{N}}}$ ($M_{\mathfrak{p}_{\mathcal{N}}}$ defined in the previous section) for the irreducible representation that describes radiation. $\mathcal{C}_{\lambda_P}$ should contain the point $(E, P) = (0, 0)$ since in the low momentum and energy limit the deformed dispersion relation should be the usual photon's dispersion relation. Consider additionally that for any other $\lambda \in M_{\mathfrak{p}_{\mathcal{N}}}$, $\mathcal{C}_\lambda$ contains the point $(E, P) = (E^{(0)}_\lambda = m^2, 0)$, i.e. we are assuming a relativistic low energy-momentum limit for every particle's deformed dispersion relation associated to $\mathfrak{p}_{\mathcal{N}}$. Since $\mathcal{C}_\lambda$ is rotationally invariant by hypothesis (a condition related to the perfect fluid description), let us consider the graph relating the momentum's magnitude to the energy in a particular irreducible representation with the dispersion relation Casimir λ: $||P|| = P_\lambda(E)$. It happens that $P_\lambda(E) \geq 0$ and there does not exist E such that $P_{\lambda_1}(E) = P_{\lambda_2}(E)$ for $\lambda_1 \neq \lambda_2$, since $C(E, P)$ is an analytic function. Therefore $P_\lambda(E^{(0)}_\lambda) = 0 < P_{\lambda_P}(E_\lambda)$, which implies that $P_\lambda(E) < P_{\lambda_P}(E)$ for all E. Indeed, if there does exist an E such that $P_\lambda(E) > P_{\lambda_P}(E)$, it follows from the continuity of $P_\lambda(E)$ (by analyticity of the Casimir $C(E, P)$) that there must exist an E such that $P_\lambda(E) = P_{\lambda_P}(E)$. But such a point cannot exist. The conclusion is that if each $\mathcal{C}_\lambda$ with positive definite energy is connected (to use the continuity argument above) and has relativistic energy in the low energy-momentum limit, a limitation of photon momentum implies a limitation of momentum for all positive definite energy irreducible representations.

The reasoning above is only valid for irreducible representations. Let us now argue about the general case. To do that, let us consider an important result in the representations of Von Neumann algebras[10] that additionally is going to be the basis of a possible new procedure to obtain the correct non-relativistic inflationary equation of state. This result comes from a powerful form of the spectral theorem. Indeed, in quantum mechanics, we know that, given an irreducible representation of the Heisenberg algebra, there exists a unitary transform U such that we can put the position or momentum operator (not at same time) as multiplicative operators in the Hilbert space $L^2(\mathbb{R})$ of square (Lebesgue) integrable functions, that is: $\hat{x}\,|\Psi\rangle = x\Psi(x)$, $\Psi(x) \in L^2(\mathbb{R})$. This diagonal form, however, is not general. It is only valid

[10] A Von Neumann algebra $\mathcal{A}$ is a subalgebra of the operators in the Hilbert space $\mathcal{B}(\mathcal{H})$ that is an $*$-algebra, that is, it contains the adjoint of each element, and is equal to its bicommutant, that is, $\mathcal{A} = \mathcal{A}''$. The bicommutant of $\mathcal{A}$ is the set of all operators in the Hilbert space that commutes with the commutant $\mathcal{A}'$, where the commutant is the set of all operafors that comute with $\mathcal{A}$.

for the so called cyclic operators[11] with spectrum equal to $\mathbb{R}$. The general form of this decomposition is the so called direct integral representation (see [32] for a proof).

Given a self adjoint operator A in some domain $\mathcal{D}$ in a separable (i.e., with countable basis) Hilbert space H, there exists a unitary transform $U : H \to \mathfrak{h}$ such that

$$\mathfrak{h} = \int^{\oplus} \mathcal{H}(\lambda) d\mu(\lambda), \tag{61}$$

$\mathcal{H}(\lambda)$ a (possibly infinite dimensional) separable Hilbert space for each λ. This is the Hilbert space whose vectors are written as

$$\Psi = \int^{\oplus} \Psi(\lambda) d\mu(\lambda), \quad \Psi(\lambda) \in \mathcal{H}(\lambda) \tag{62}$$

and the inner product is:

$$(\Phi, \Psi) = \int \langle \Phi(\lambda), \Psi(\lambda) \rangle_{\lambda} \, d\mu(\lambda), \tag{63}$$

$\langle \Phi(\lambda), \Psi(\lambda) \rangle_{\lambda}$ being the inner product of the Hilbert space $\mathcal{H}(\lambda)$, $\mu(\lambda)$ a suitable integration measure[12]. Furthermore, we have:

$$UAU^{-1} \int^{\oplus} \Psi(\lambda) d\mu(\lambda) = \int^{\oplus} \lambda \Psi(\lambda) d\mu(\lambda). \tag{64}$$

The domain of the operator UAU^{-1}, $U(\mathcal{D})$, is the set:

$$\int ||\lambda \Psi(\lambda)||_{\lambda}^{2} d\mu(\lambda) < \infty, \tag{65}$$

$|| * ||_{\lambda}$ being the norm of the Hilbert space $\mathcal{H}(\lambda)$.

We can do the same thing with a set of commuting self-adjoint operators A_i with a common domain $\mathcal{D}$ [13], defining the unitary transform (This is actually the Von Neumann's theorem for Abelian Von Neumann algebras):

$$U_{\mathcal{D}} : \mathcal{H} \to \mathfrak{h} = \int^{\oplus} \mathcal{H}(\lambda_1, \lambda_2, \cdots) d\mu(\lambda_1, \lambda_2, \cdots), \tag{66}$$

such that $UA_iU^{-1} \int^{\oplus} \psi(\lambda_1, \cdots \lambda_N) d\mu(\lambda_1, \cdots \lambda_N) = \int^{\oplus} \lambda_i \psi(\lambda_1, \cdots \lambda_N) d\mu(\lambda_1, \cdots \lambda_N)$.

We can define the operator X in $\mathfrak{h}$ by defining the operator $X(\lambda)$ in some domain $\mathcal{D}(\lambda)$ in each $\mathcal{H}(\lambda)$:

$$X\Psi = \int^{\oplus} X(\lambda) \Psi(\lambda) d\mu(\lambda). \tag{67}$$

We denote:

$$X = \int^{\oplus} X(\lambda) d\mu(\lambda) \tag{68}$$

as the operator whose domain is the set $\int^{\oplus} \Psi(\lambda) d\mu(\lambda)$, $\Psi(\lambda) \in \mathcal{D}(\lambda)$ and $\int ||X(\lambda)\Psi(\lambda)||_{\lambda}^{2} d\mu(\lambda) < \infty$. It is possible to prove that the algebraic function $f[X]$ of X can be writen as:

$$f[X] = \int^{\oplus} f[X(\lambda)] d\mu(\lambda), \tag{69}$$

which shows the algebraic independence of the operators defined in each $\mathcal{H}(\lambda)$.

Now suppose a representation of $\mathfrak{p}_{\mathcal{N}}$ in the Hilbert space H and $C_1, C_2, \cdots, C_N$ is the set of Casimir operators of the representation. By the uniform transform we put H in a direct integral form such that all Casimir operators are multiplicative:

$$\mathfrak{h} = \int^{\oplus} \mathcal{H}(\lambda) d\mu(\lambda), \tag{70}$$

where λ is the entire set of Casimir eigenvalues. But in each $\mathcal{H}(\lambda)$, the Casimir is a multiple of the identity, therefore, by Schur's lemma, there must exist an irreducible representation of $\mathfrak{p}_{\mathcal{N}}$ on each $\mathcal{H}(\lambda)$ whose generators are written in the form (68).

We observe that the set of Casimir operators can include operators that are not predicted by the algebra of $\mathfrak{p}_{\mathcal{N}}$, that is, other than (8), in the case of Poincaré Lie algebra. Indeed, suppose $\pi : \mathfrak{p}_{\mathcal{N}} \to \mathcal{O}(\mathcal{D})$ ($\mathcal{O}(\mathcal{D})$ stands for operators in the domain $\mathcal{D}$) is a irreducible representation, and the algebra predicts a particular Casimir set $\mathcal{C}_1, \mathcal{C}_2, \cdots$ written as algebraic functions of the generators of $\mathfrak{p}_{\mathcal{N}}$. We can write another representation in terms of the irreducible one: $\pi' = \pi \oplus \pi$. This is a reducible representation, but there doues not exist another Casimir in the image of the representation π' other than $\pi'(\mathcal{C}_i)$. However, $C'' = (a\pi[\mathcal{C}_i]) \oplus (b\pi[\mathcal{C}_i])$, $a, b \in \mathbb{R}$, $a \neq b$, is a Casimir operator that is not a multiple of the identity and it is not in the image of the original representation. This is because the representation of an algebra like the Lie algebra (actually its enveloping algebra) is not a Von Neumann algebra (see footnote (10)) such that $\mathcal{A} = \mathcal{A}''$, that is, that contains all its Casimir elements. It is contained in some Von Neumann algebra. Those additional Casimirs are related to the degenerance of the representation, i.e., the number of times, finite or infinite, that it appears in the decomposition.

We then represent this decomposition as:

$$\mathfrak{h} = \int^{\oplus} \mathcal{H}(\lambda, \sigma) d\mu(\lambda, \sigma), \tag{71}$$

where λ is the algebraic Casimir eigenvalues, and σ is related to the desgenerance.

Now suppose that in the decomposition of the representation of $\mathfrak{p}_{\mathcal{N}}$ there appears irreducible representations π_{λ} with dispersion relation eigenvalues $\lambda_1 \notin M_{\mathfrak{p}_{\mathcal{N}}}$

[11] A selfadjoint operator A in some domain $\mathcal{D}$ in the Hilbert space $\mathcal{H}$ is cyclic if there exists some $\Psi \in \mathcal{D}$ such that the set $A^n \Psi$, for all n, spans the entire Hilbert space

[12] There is an additional detail that there exists a countable set of mensurable sets X_i of λ values such that the dimension of the Hilbert space $\mathcal{H}(\lambda)$ is the same for the same X_i and, when it is an infinite dimensional Hilbert space for some X_i, it is a separable Hilbert space.

[13] Actually, A_i is required to be essentially self-adjoint in the domain $\mathcal{D}$

for some set B of λ with non-zero measure. Suppose also there is a mensurable function $f(\lambda)$ such that $\int d\mu(\lambda)|f(\lambda)|^2 < \infty$. If B has a finite measure, this function can be chosen as the characteristic function of B, χ_B, which is equal to unity in B and zero otherwise. We can choose a normalized vector ψ_λ of negative energy E_λ in each $\mathcal{H}(\lambda)$ with $\lambda \in B$ and the null vector in every other λ value. The state $\int^{\oplus} d\mu(\lambda)f(\lambda)\psi_\lambda$ belongs to the Hilbert space $\mathfrak{h}$ and has negative energy. The conclusion is that for all direct integral decompositions of $\mathfrak{p}_{\mathcal{N}}$ with positive definite energy, only irreducible representations with $\lambda_0 \in M_{\mathfrak{p}_{\mathcal{N}}}$ do appear (except by zero measure sets).

Furthermore, we can write P^μ as a multiplicative operator if we apply a further unitary transform in $\mathfrak{h}$ that represents each $\mathcal{H}(\lambda,\sigma)$ in (71) as the Hilbert space of N component functions[14] defined in the place of the points $(E,\vec{P}) \in \mathbb{R}^4$ in the energy-momentum space that satisfies $C(E,P) = \lambda_1$ with measure ν[15] (i.e. $(E,P) \in \mathcal{C}_{\lambda_1}$, recalling that $\lambda = (\lambda_1, \lambda_2, \cdots)$). Combining the measures in $\mathcal{C}_{\lambda_1}$ and λ_1 [16] , we define a new measure μ' in four-momentum space and write:

$$\mathfrak{h} = \int^{\oplus} \mathcal{H}(p^\mu, \lambda', \sigma) d\mu'(p^\mu) d\nu(\lambda', \sigma), \tag{72}$$

where λ' denotes the remaing algebraic Casimir elements $\lambda' = (\lambda_2, \cdots)$. The support of the measure μ', i.e., the region giving contributions to the integral[17] , is the set of points belonging to $\mathcal{C}_\lambda$ for each λ belonging to the direct integral decomposition. For simplicity of notation, let us denote

$$\mathfrak{h} = \int^{\oplus} \mathcal{H}(p^\mu, \tau) d\chi, \tag{73}$$

where $\tau = (\lambda', \sigma)$.

We have the operator P^μ defined in $\mathfrak{h}$ by the operator ${P^\mu}_{(p^\mu,\tau)}$ in each $\mathcal{H}(p^\mu,\tau)$ such that, for all $\Psi_{(p^\mu,\tau)} \in \mathcal{H}(p^\mu,\tau)$:

$$P^\mu_{(p^\mu,\sigma)} \Psi_{(p^\mu,\sigma)} = p^\mu \Psi_{(p^\mu,\sigma)}, \quad p^\mu \in V^+_{\mathfrak{p}_{\mathcal{N}}} \subseteq \mathbb{R}^4. \tag{74}$$

Then, for all normalized $\Psi \in \mathfrak{h}$, $\langle\Psi| P^\mu |\Psi\rangle \in \overline{V^+_{\mathfrak{p}_{\mathcal{N}}}}_c$, where $\overline{V^+_{\mathfrak{p}_{\mathcal{N}}}}_c$ denotes the convex linear closure [18] .

Indeed, we can partition the set $V^+_{\mathfrak{p}_{\mathcal{N}}}$ in regions $U_n = V^+_{\mathfrak{p}_{\mathcal{N}}} \cap R_{ijkl}$, where n is a set of indexes $n = (i,j,k,n)$ and R_{ijkl} are the rectangles:

$$R_{ijkl} = [p^0_i, p^0_{i+1}) \times [p^1_j, p^1_{j+1}) \times [p^2_k, p^2_{k+1}) \times [p^3_l, p^3_{l+1}), \tag{75}$$

obtained from partitions p^μ_i, $i = 1,2,\cdots N$, $\mu = 0,1,2,3$, such that $\bigcup_n R_n \supset V$.

It follows that $\langle\Psi| P^\mu |\Psi\rangle = \int d\chi p^\mu ||\psi(p^\mu,\tau)||^2$, which implies the inequalities:

$$\sum_n \inf_{U_n}(p^\mu) \int_{U_n} ||\psi(p^\mu,\tau)||^2 d\chi \leq \langle\Psi| P^\mu |\Psi\rangle \tag{76}$$

$$\langle\Psi| P^\mu |\Psi\rangle \leq \sum_n \sup_{U_n}(p^\mu) \int_{U_n} ||\psi(p^\mu,\tau)||^2 d\chi, \tag{77}$$

but $||\psi(p^\mu,\tau)||^2$ is positive definite and $\int_{V^+_{\mathfrak{p}_{\mathcal{N}}}} d\chi ||\psi(p^\mu,\tau)||^2 = 1$, in such a way that both inequalities can be written:

$$\sum_n p^\mu_n c_n, \quad \sum_n c_n = 1 \text{ with } p^\mu_n \in V^+_{\mathfrak{p}_{\mathcal{N}}} \text{ and } c_n \geq 0. \tag{78}$$

Suppose that $\sum_{n=0}^N p^\mu_n c_n \in \overline{V^+_{\mathfrak{p}_{\mathcal{N}}}}_c$ for $\sum_{n=0}^N c_n = 1$, then $\sum_{n=0}^{N+1} p^\mu_n c'_n \in \overline{V^+_{\mathfrak{p}_{\mathcal{N}}}}_c$ for $\sum_{n=0}^{N+1} c'_n = 1$. Indeed, we can write

$$\sum_{n=0}^{N+1} p^\mu_n c'_n = \frac{\sum_{n=0}^N p^\mu_n c'_n}{\sum_{n=0}^N c'_n} \frac{\sum_{n=0}^N c'_n}{\sum_{n=0}^{N+1} c'_n} + p^\mu_{N+1}\left(1 - \frac{\sum_{n=0}^N c'_n}{\sum_{n=0}^{N+1} c'_n}\right) \tag{79}$$

But for $N = 1$ we have $p^\mu_0 c_0 + p^\mu_1(1 - c_0) \in \overline{V^+_{\mathfrak{p}_{\mathcal{N}}}}_c$.

Therefore, if $p^\mu \in V^+_{\mathfrak{p}_{\mathcal{N}}}$ for every (p^μ,σ), $||\vec{p}|| < p_{\max}$, it follows that for every representation of $\mathfrak{p}_{\mathcal{N}}$ with positive definite energy, the moment is limited by photon momentum:

$$\langle\Psi| P^j |\Psi\rangle \leq p^j_{\max}, \tag{80}$$

since every convex linear combination with limited momentum has the same bound to the momentum.

But the generators associated to representation (53) are of the form:

$$X_F = \sum_{N=1}^{\infty} {}^{\oplus} (X \otimes I \cdots \otimes I + I \otimes X \otimes \cdots \otimes I + \cdots + I \otimes I \cdots \otimes X)_N \tag{81}$$

In particular, the energy of this representation is positive definite and the moment is unbounded (it assumes every integer multiple of the irreducible representation momentum). Therefore, the generators of (53) do not realize $\mathfrak{p}_{\mathcal{N}}$. Besides, since every representation of a Lie group induces a representation of the same Lie algebra and (53) is a representation of the group $\mathcal{P}_{\mathcal{N}}$, there is no Lie group that realizes non-commutative inflation.

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[14] As stated, this decomposition is valid for irreducible representations if the associated Lie algebra has space-time translation generators that form an ideal

[15] That is, the inner product of the irreducible representation is $(\Phi,\Psi) = \int_{\mathcal{C}_\lambda} \psi^*(p^\mu)\psi(p^\mu)d\nu = \int_{\mathcal{C}_\lambda} \sum_\sigma \psi^*_\sigma(p^\mu)\psi_\sigma(p^\mu)d\nu$, where ψ_σ are the components of ψ.

[16] If we have two measurable sets X_1 and X_2 with measures ν_1 and ν_2, we can define a new measurable set $X_1 \times X_2$ in which all measurable sets are of the Cartesian product form $A_1 \times A_2$ for A_i a measurable set of X_i and the measure of $A_1 \times A_2$ is $\nu_1(A_1) \cdot \nu_2(A_2)$. The measure of (70) is the Cartesian product form

[17] the largest closed set in which every open neighborhood has positive measure

[18] The set of all points written in the form $\alpha p^\mu_1 + (1-\alpha) p^\mu_2$, $\alpha \in [0,1]$ e $p^\mu_1, p^\mu_2 \in V^+_{\mathfrak{p}_{\mathcal{N}}}$. $\overline{V^+_{\mathfrak{p}_{\mathcal{N}}}}_c$ is obtained joining to $V^+_{\mathfrak{p}_{\mathcal{N}}}$ every line segment that joints any two points of $V^+_{\mathfrak{p}_{\mathcal{N}}}$.

VII. THE ALTERNATIVE PRESCRIPTION FOR THE NON-RELATIVISTIC INFLATIONARY EQUATION OF STATE

The conclusion of the previous section is that, given the condition for minimum duration of inflation in non-commutative inflation [15], which implies that the momentum of an individual photon is limited, and assuming that all C_λ (the mass shell deformation) with positive definite energy is connected and allows a zero momentum state, the non-commutative inflation is then necessarily associated with a non-trivial Hopf algebra structure with a non-trivial coproduct structure. Therefore Eq. (53), which leads to the fundamental equations of non-commutative inflation, cannot be the appropriate time-translation generator, according to the discussion of section (V). Considering the group point of view, that is, considering that we are representing the group $\mathcal{P}_\mathcal{N}$ whose generators are e^{-iX_it} for $X_i \in \mathfrak{p}_\mathcal{N}$, to define the physics, we are restricting the possible representations that are used in such a construction, excluding representation (53). This kind of restriction is not new in physics, since we already restrict the representations of the Poincaré group that can appear in a relativistic quantum theory to those of positive definite energy, for example.

If we are able to find a Hopf algebra deformation $\mathfrak{p}_\mathcal{N}$ satisfying all the requirements of section (V) and, additionally, a coproduct $\Delta^{\mathfrak{p}_\mathcal{N}}$ (defined in section (III)) satisfying:

$$\begin{aligned}\left(\Psi_{p^\mu_{\max}} \otimes \Phi_{p^\mu_{\max}}, \Delta^{\mathfrak{p}_\mathcal{N}} X_i^{\mathfrak{p}_\mathcal{N}} \Psi_{p^\mu_{\max}} \otimes \Phi_{p^\mu_{\max}}\right) \\ \to \left(\Psi_{p^\mu_{\max}} \otimes \Phi_{p^\mu_{\max}}, \Delta^{\mathfrak{p}} X_i^{\mathfrak{p}_\mathcal{N}} \Psi_{p^\mu_{\max}} \otimes \Phi_{p^\mu_{\max}}\right),\end{aligned} \quad (82)$$

where $\Psi_{p^\mu_{\max}}$ and $\Phi_{p^\mu_{\max}}$ are normalized states written as in Eq. (58) and the limit is when $(E,P) \to (0,0)$. $X_i^{\mathfrak{p}_\mathcal{N}}$ is the Hopf algebra deformation of the generator $X_i^{\mathfrak{p}}$ of the Poincaré Lie algebra and $\Delta^{\mathfrak{p}}$ is the Lie algebra coproduct: $\Delta^{\mathfrak{p}} X = X \otimes I + I \otimes X$. This rule means that the new coproduct $\Delta^{\mathfrak{p}_\mathcal{N}}$ becomes the usual coproduct of Lie algebras for low energy and momentum.

Given such a Hopf algebra deformation, with the appropriate coproduct, we can define the time translation generator that replaces Eq. (53) and is part of a representation of $\mathfrak{p}_\mathcal{N}$:

$$H_F^{\mathfrak{p}_\mathcal{N}} = \sum_{N=0}^{\infty}{}^{\oplus} \Delta^N H, \quad (83)$$

where Δ^N is defined in Eq.(41). In some sense, it implies that particles interact even in the free theory, that is, they do not evolve in time in a completely independent way.

That prescription has the disadvantage that we can only apply it to deform the free fields. Besides that, we need a full definition of the algebra and Hopf algebra structure associated to a particular photon phenomenological dispersion relation, which includes, in particular, the assurance of the existence of a Hopf algebra associated to it.

It is however easy to assure the existence of a algebra (without considering its coalgebraic structure, i.e., its coproduct, counity and antipode) that satisfies all the requirements discussed in section (V) and leads to a particular photon dispersion relation[19] . We can use an alternative prescription only based on algebra properties that can be applied to interacting field theory and free fields and assuring that we are constructing a physics that is based on representations of a group $\mathcal{P}_\mathcal{N}$ that realizes the same algebra of generators $\mathfrak{p}_\mathcal{N}$.

The basic idea is to explore the fact that every relativistic quantum field theory, associated to some Lagrangian density, has associated with it a representation of the Poincaré Lie group in the positive definite Hilbert space (i.e. such that the norm of states is positive). This representation enters in the relativistic covariance law of the fields:

$$A^i{}_j[(\Lambda, a)]\phi_i(\Lambda x + a) = U[(\Lambda, a)]\phi_j(x)U^{-1}[(\Lambda, a)] \quad (85)$$

where $A^i{}_j[(\Lambda, a)]$ is a finite dimensional representation of the Poincaré Lie group. It happens that (85) may not be satisfied in the positive definite Hilbert space, as in the case of electrodynamics, but we can restrict the action of $U[(\Lambda, a)]$ to the equivalence class of physical states (something similar to what we have done in the GNS construction to construc equivalence classes that describe the same state).

This representation can be recovered from the Wightman functions by the so called Wightman reconstruction theorem, that is an application of the GNS construction. Alternatively, we can obtain it by other classes of functions like the time-ordered Green's functions or the Schwinger functions (see some original papers on this subject [33], [34] and [35]). In the case of free fields, we already have that representation, it is simply (15) for a suitable choice of irreducible representation of Poincaré group.

We then put the associated representation in the direct integral form:

$$\mathcal{U}^\mathcal{P} = \mathcal{U}_0 \oplus \int^{\oplus} d\mu(\lambda, \sigma)\mathcal{U}^\mathcal{P}_{\lambda,\sigma}, \quad (86)$$

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[19] In fact, consider X_i the infinitesimal generators of the scalar representation of the homogeneous Lorentz group in momentum space:

$$\phi(x^\mu) \to \phi(\Lambda x^\mu) \quad (84)$$

They are contravariant vectors, or first order differential operators. Consider the diffeomorphism $\Phi : (E,p) \to (\bar{E}, \bar{p})$ given by $\bar{E} = E$; $\bar{p} = p/f(E)$, the $f(E)$ given in (27). Define the new algebra as multiplication operators E, p and $\bar{X}_i = \Phi_* X_i$, the push-foward operator $\Phi_* X_i \triangleright f = X_i \triangleright f \circ \Phi$. This new algebra satisfy $[\bar{E}, \bar{p}] = 0$; $[\bar{X}_i, \bar{X}_j] = C^k_{ij}\bar{X}_k$, C^k_{ij} the same structure constants of the original Lorentz group, but $[\bar{X}_i, E] = (\bar{X}_i \triangleright E) = F(E,p)$ and $[\bar{X}_i, p] = (\bar{X}_i \triangleright p) = G(E,p)$; As required $[C(E,p), E] = [C(E,p), p] = [C(E,p), \bar{X}_i] = 0$, where $C(E,p) = f^2(E)p^2 - E^2$. All the conditions of the section (V) are satisfied

that is defined in:

$$\mathcal{H}^{\mathcal{P}} = c\Psi_0 \oplus \int^{\oplus} d\mu(\lambda,\sigma)\mathcal{H}^{\mathcal{P}}_{\lambda,\sigma}, \quad (87)$$

where $\mathcal{U}_0$ is the trivial vacuum representation, that leaves the vacuum subspace $c\Psi_0$ invariant, while λ denotes the eigenvalues of those Casimir elements predicted by the Poincaré Lie algebra (8) and σ denotes the Casimir eingenvalues related to degenerance. That is, the irreducible representation $\mathcal{U}^{\mathcal{P}}_{\lambda,\sigma}$ is the same for all σ and we have a σ value for each time that the same representation appears in the decomposition.

We then postulate:

Prescription 2 ($\pi_{\mathcal{A}}$)**.** *Be $\mathfrak{p}_{\mathcal{N}}$ an algebra (or Hopf algebra) with generators X_i and $\mathcal{P}_{\mathcal{N}}$ the group generated by one parameter subgroups of the form $e^{-iX_it_i}$, $t_i \in \mathbb{R}$. Be $\pi_{\lambda'} : \mathfrak{p}_{\mathcal{N}} \to O(\mathcal{D})$ the irreducible representation of $\mathfrak{p}_{\mathcal{N}}$ in Hilbert space (obtained by GNS construction with states in $\mathfrak{p}_{\mathcal{N}}$ or $\mathfrak{p}^*_{\mathcal{N}}$) associated to algebraic Casimir eingenvalues $\lambda' = (\lambda_1, \lambda_i, \cdots)$. Be $\mathcal{U}^{\mathcal{P}_{\mathcal{N}}}_{\lambda'}$ the representation of $\mathcal{P}_{\mathcal{N}}$ generated by $e^{-i\pi_{\lambda'}(X_i)t_i}$. The representation of $\mathcal{P}_{\mathcal{N}}$ that deforms a relativistic quantum field theory is obtained by the associated Poincaré Lie group representation (restricted to positive definite Hilbert space of physical states) put in the direct integral form:*

$$\mathcal{U}^{\mathcal{P}} = \mathcal{U}_0 \oplus \int^{\oplus} d\mu(\lambda,\sigma)\mathcal{U}^{\mathcal{P}}_{\lambda,\sigma}, \quad (88)$$

by the replacement:

$$\mathcal{U}^{\mathcal{P}}_{\lambda} \to \mathcal{U}^{\mathcal{P}_{\mathcal{N}}}_{\lambda'}, \quad (89)$$

for the set of Casimir eigenvalues λ' satisfying the condition:

$$\Big(\mathfrak{u}(\Psi^{\lambda}_{p^{\mu}_{\max}}), e^{-i\pi_{\lambda'}[X^{\mathfrak{p}_{\mathcal{N}}}_i]t_i}\mathfrak{u}(\Phi^{\lambda}_{p^{\mu}_{\max}})\Big) \to \Big(\Psi^{\lambda}_{p^{\mu}_{\max}}, e^{-i\pi_{\lambda}[X^{\mathfrak{p}}_i]t_i}\Phi^{\lambda}_{p^{\mu}_{\max}}\Big). \quad (90)$$

when $(E_{max}, p^j_{max}) \to (m, 0)$ for normalized states $\Psi^{\lambda}_{p^{\mu}_{\max}}$ and $\Phi^{\lambda}_{p^{\mu}_{\max}}$ written in the form (58).

Here, $\mathfrak{u} : \mathcal{H}_1 \to \mathcal{H}_2$ defines an unitary transform connecting the Hilbert spaces where the representations of the Poncaré group and $\mathcal{P}_{\mathcal{N}}$ are defined. As before, $X^{\mathfrak{p}_{\mathcal{N}}}$ is the corresponding operator in $\mathfrak{p}_{\mathcal{N}}$ to the Poincaré Lie algebra element $X^{\mathfrak{p}}$. m is the relativistic energy at zero momentum that, by hypothesis, is the same for the corresponding $\mathcal{P}_{\mathcal{N}}$ representation.

This prescription defines the new time translation generator in the direct integral form:

$$H^{\mathfrak{p}_{\mathcal{N}}} = 0 \oplus \int^{\oplus} d\mu(\lambda,\sigma)H^{\mathfrak{p}_{\mathcal{N}}}_{\lambda',\sigma}, \quad (91)$$

where $H^{\mathfrak{p}_{\mathcal{N}}}_{\lambda',\sigma}$ is the Hamiltonian of the irreducible representation of $\mathfrak{p}_{\mathcal{N}}$ associated to the set of eigenvalues λ' (that is diagonal in the energy-momentum representation and related to the momentum by $C(H^{\mathfrak{p}_{\mathcal{N}}}_{\lambda,\sigma}, P) = \lambda_1$). That is, not only the photon dispersion relation enters in the Hamiltonian of the theory but the dispersion relations of others species of particles.

The condition (90) is actually stronger than (59) and allows us to assure the convergence of the $\mathcal{P}_{\mathcal{N}}$ representation to the original relativistic representation in the weak sense (i.e., in the sense of matrix elements: $\big(\mathfrak{u}(\Psi), \mathcal{U}^{\mathcal{N}\mathcal{P}}\mathfrak{u}(\Phi)\big) \to \big(\Psi, \mathcal{U}^{\mathcal{P}}\Phi\big)$, where $\mathfrak{u}$ is a unitary map that connects the Hilbert spaces where the Poincaré representation is defined to the one where the $\mathcal{P}_{\mathcal{N}}$ representation is defined) if all intermediate states of direct integral decomposition converge as (58). This corresponds to states of sufficiently low energy and momentum [20] .

VIII. CONCLUSION

In a previous paper [15], we showed that the fundamental equations of non-commutative inflation, first obtained in [19] and applied in a model of inflation in [1], can actually be related to a representation of some group that deforms Poincaré Lie group, a group that can be constructed by starting from a Hopf algebra (allowing a possible connection with the original motivation of non-commutative spaces, along the lines of [13]). We showed that the condition of minimum duration of inflation is actually related to a limitation of the momentum of an individual photon. In the current paper, we showed that a group that replaces Poincaré must satisty some important physical constraints already satisfied by the Poincaré Lie group. These constraints do not depend on cosmological requirements and affect not only the group's abstract algebraic properties, but the representations themselves. Considering these constraints, the limitation of the photon's momentum, together with the minimal additional conditions on the deformed mass shell, such as to be connected at least for those solutions of positive definite energy and the same solutions allowing a zero momentum state, implies that the notion of Fock space is not suitable and a Hamiltonian like (17) cannot be the starting point of the fundamental equations of the non-commutative inflation.

Everything starts by the fact that we are replacing the Poincaré group $\mathcal{P}$ by a group $\mathcal{P}_{\mathcal{N}}$ that has connected one-parameter subgroups that allow us, by Wigner and

[20] Lebesgue's theorem says that if $g_N(x) \to g(x)$ for all x and $|g_N(x)| \leq f(x)$ for some integrable $f(x)$, then $g(x)$ is Lebesgue integrable and $\int g_N(x) \to \int g(x)$. Since we know that, for the original QFT, there exists Hilbert space vectors $\Psi_{\sigma,\lambda}$ in each $\mathcal{H}_{\lambda,\sigma}$ such that the inner product integral converges, and the unitary transformation satisfies $\left|\Big(\mathfrak{u}(\Psi_{\lambda,\sigma}), \mathcal{U}^{\mathcal{N}\mathcal{P}}_{\lambda',\sigma'}\mathfrak{u}(\Phi_{\lambda,\sigma})\Big)\right| \leq ||\Psi_{\lambda,\sigma}|| \cdot ||\Phi_{\lambda,\sigma}|| \leq ||\Phi_{\lambda,\sigma}||^2 + ||\Psi_{\lambda,\sigma}||^2$ which is a measurable function with finite integral, we have the fulfillment of conditions of the Lebesgue theorem.

Stone theorems, to define generators of representations as self-adjoint operators that we assume to form an algebra $\mathfrak{p}_{\mathcal{N}}$ with appropriate physical properties. Essentially, we show that, although a Fock space representation (53) is a valid mathematical representation for $\mathcal{P}_{\mathcal{N}}$, for the kind of deformed dispersion relation that drives inflation with minimum duration, this representation does not satisfy the same algebra of generators used in the irreducible representations. As a consequence, non-commutative inflation is not related to some Lie group, since every representation of a Lie group induces a representation of the very same Lie algebra of generators.

As we argue, the use of this Fock like representation is a problem for a physical theory, since it might imply that a stable state in a particular reference frame is an unstable state in another one, or a possible S matrix connecting a one particle state with a n-particle state does not conserve energy in all reference frames. In particular, we do not have a unique well defined notion of a dispersion relation that replaces the usual relativistic one.

It does not mean that we cannot use a group like $\mathcal{P}_{\mathcal{N}}$ in physics. It means that we must restrict the class of representations of $\mathcal{P}_{\mathcal{N}}$ that is used to construct physics, something similar to restricting the representations of the Poincaré group that appear in QFT to those with positive definite energy. But now, besides that, we must restrict the representations of the $\mathcal{P}_{\mathcal{N}}$ group to those with positive definite energy and that induces a representation of the same algebra of generators $\mathfrak{p}_{\mathcal{N}}$.

An alternative way to say that (53) is not a suitable representation of $\mathcal{P}_{\mathcal{N}}$ is to say that (17) is not a time-translation generator for $\mathfrak{p}_{\mathcal{N}}$ if H is. Since, in the Hopf algebra language, (17) is the coproduct rule for Linear Lie algebras, this suggests to us that non-commutative inflation is actually related to some non-trivial Hopf algebra, understanding that a Hopf algebra is an algebra with a rule to construct tensor products that satisfy the same algebra.

The Hopf algebra concept is a possible link between the non-commutative space concept and non-commutative inflation, since an alternative interpretation for Hopf algebras is that it is a set of transformations that acts on other algebraic structures in a way dependent on the product rule. The product rule is the essential information of non-commutative spaces (see [17]). We can therefore use the coproduct rule of Hopf algebra to define the suitable non-commutative inflationary Hamiltonian that defines the canonical partition function that describes the radiation thermodynamics as in (83). This prescription needs the full knowledge of the Hopf algebra structure associated to some phenomenological dispersion relation, a problem by itself, and can only be applied to free fields.

We can consider an alternative prescription that is based only on algebraic aspects, that is, it does not depend on the coalgebraic aspects, which means that it does not depend on the information that differs the Hopf algebra from a simple algebra, as the coproduct. This prescription is based on the infinite dimensional generalization of the concept of direct sum, the direct integral. This prescription was originally proposed by us in [15], as an alternative to cover the interacting case and give a qualitative account of the behavior of the interacting non-commutative inflation. This prescription is essentially an application of an important Von Neumann theorem about the general representation of Von Neumann algebras. This prescription allows us to define a new Hamiltonian for non-commutative inflation that is indeed a time-translation generator of the very same algebra of generators that defines the dispersion relation Casimir. Besides that, it assures to recover the corresponding Poincaré representation at low energy and momentum. The new Hamiltonian (91) is a function not only of the photon's dispersion relation but a function of the dispersion relation of other particles of the theory.

To obtain the new Hamiltonian for non-commutative inflation, we need the direct integral decomposition of the representation (15), that is a solved problem in the literature since the early days of QFT (for example, [36], [37]). We will address the problem of obtaining the new equations of state in a future paper. The main consequence of our results is that all the work previously done on non-commutative inflation must be redone, that is, finding a new equation of state, finding inflationary dispersion relations that assure successful inflation, etc. In particular, the use of the same class of dispersion relations of the original equations of the model will imply the limitation of the momentum of the theory, even the macroscopic one. We can, however, use the same rule to construct the Hamiltonian even if our theory does not have that limitation of momentum.

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